Generalization of Mittag-Leffler function andit's Application in Quantum-Calculus

Alok Jain
Assistant Professor, Department of Mathemetics,
Amity University, Gwalior M.P. (India)
E-Mail:- ajain1@gwa.amity.edu

ABSTRACT
In the present paper, the authors derived the basic analogue of Mittag-Leffler function with the applications of q-beta function and obtain some important results of Mittag-Leffler functionin terms of generalized wright function.

Keywords: q-Beta function, fractional q-operator; Basic analogue of the Mittag-Leffler function.

INTRODUCTION
During the second half of the twentieth century, considerable amount of research in fractional calculus was published in engineering literature. Indeed, recent advances of fractional calculus are dominated by modern examples of applications in differential and integral equations, physics, signal processing, fluid mechanics, visco elasticity, mathematical biology, and electrochemistry. There is no doubt that fractional calculus has become an exciting new mathematical tool for the solution of diverse problems in mathematics, science, and engineering. Inspired by the great success of fractional calculus many research workers, mathematician concentrated on another dimension of calculus which is sometimes called calculus without limits or more popularly q-calculus. The q-calculus was initiated in twenties of the last century. Kac and Cheung's book [1] entitled "Quantum Calculus" provides the basics of such type of calculus. The fractional q-calculus is the q-extension of the ordinary fractional calculus. The present paper deals with the investigations of q-integrals and q-derivatives of arbitrary order, of q-Mittag-Leffler.

Definitions and preliminaries in this paper:
1. **Mittag-Leffler Function:** The function $E_\mu(z)$ is defined by the series representation

$E_\mu(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\mu n+1)}$, μ>0, t ∈C.

Mittag-Leffler [2], Wiman [3] and Agarwal [1] investigated the generalization of the above function $\boldsymbol{E_\upsilon}(z)$ in the following manner [3].

$E_{\upsilon,\rho}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\upsilon n+\rho)}$, υ>0,ρ>0, t ∈C,where C is the set of complex numbers.

A more generalized form of Mittag-Leffler function is introduced by Prabhakar [54] as
$E_{\upsilon,\rho}^{\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\upsilon n+\rho) n!}$.

The generalized Fox-Wright function ${}_p\psi_q(z)$ defined for z ∈C, a_i , b_j ∈C and α_i, β_j ∈R
$\{\beta_j \neq 0, i = 1,2, \dots p, j = 1,2,3, \dots q\}$ is given by the series

$$ {}_p\psi_q(t) = {}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q}\end{matrix}\ ,\ t\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i s)t^n}{\prod_j^q\Gamma(b_j+\beta_j s)\quad n!} $$

2. Riemann-Liouville q-fractional Operator

Agarwal [2], introduced the q-analogue of the Reimann-Liouville fractional integral operator as follows.

$$ I_{q,x}^{\alpha}f(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)_{\alpha-1}f(t)d_q(t)\ ;\ Re(\alpha) > 0. $$

In particular, for f(x) $=x^p$, we have

$$ I_{q,x}^{\alpha}(x^p) = \frac{\Gamma_q(p+1)}{\Gamma_q(p+\alpha+1)}x^{p+\alpha};\ Re(\alpha) > 0 $$

Also q-analogue of the Reimann-Liouville fractional derivative definedas

$$ D_{q,x}^{\alpha}f(x) = D_q^n(I_{q,x}^{n-\alpha}f)x;\ Re(\alpha) < 0, \text{ and } |q|<1. $$

in particular, for f(x) = x^p, we have

$$ D_{q,x}^{\alpha}(x^p) = \frac{\Gamma_q(p+1)}{\Gamma_q(p-\alpha+1)}x^{p-\alpha};\ Re(\alpha) < 0, |q|<1. $$

Main Results:

In this section, we have introduced the q- analogue of Mittag-Leffler function, which was first time coined by Mittag-Leffler in the year (1903) in [3]. It is defined as follows

$$ E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)};\quad Re(\alpha) > 0 $$

Originally Mittag-Leffler considered only the parameter α and assumed it as positive, but later on the generalization with two complex parameters was considered by Wiman as

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)};\ Re(\alpha) > 0;\ \text{and } \beta \in C. $$

Generally, $E_{\alpha,1}(z) = E_\alpha(z)$

In 1971, Prabhakar [54] introduced the more generalized function$E^{\gamma}_{\alpha,\beta}(z)$defined as follows:

$$ E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)};\quad \text{for } Re(\alpha) > 0,\ Re(\beta) > 0, Re(\gamma) > 0;\ \text{and } \alpha,\beta,\gamma \in C. $$

Fox-Wright Generalized Hypergeometric Functions: The Fox–Wright function (also known as Fox–Wright Psi function or just Wright function) is a generalization of the generalized hypergeometric function ${}_pF_q(z)$ based on an idea of E. Maitland Wright (1935)[7].This is defined as

$$ {}_p\Psi_q\left[\left(\begin{matrix}(a_1,A_1)(a_2,A_2)\ldots(a_p,A_p) \\ (b_1,B_1)(b_2,B_2)\ldots(b_q,B_q)\end{matrix}\middle| z\right)\right] = \sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\ldots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\ldots\Gamma(b_q+nB_q)}\frac{z^n}{n!}. $$

In the sequel to thisstudy, we have introduced the basic analogue of Mittag-Leffler as follows

$$ E_\alpha(z:q) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma_q(\alpha n+1)};\quad Re(\alpha) > 0 $$

Thefunction$E_\alpha(z:q$ turns out to be a special case of basic analogue of H- function. Therefore it converges under the convergence conditions of basic analogue of H- function which are as follows. The integral converges if $Re[s\log(z) - \log\sin\pi s] < 0$, on the contour C, where 0<|q|< 1,as verified by Saxena, et al [6].

Theorem (1):

Let α >0, t $\geq 0, \mu \geq 0, Re(\mu - v) > 0$ and |q|< 1, α, $\mu, v \in C$. let $\mathbf{I}_{q,x}^{\alpha}$ be the Riemann- Liouville fractional integral operator, then there holds following relations

$$\mathbf{I}_q^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = x^{\gamma+\alpha-1}\ {}_2\Psi_1\left(\begin{matrix}(\gamma,1)(1,1)\\(\gamma+\alpha+1,\mu)\end{matrix}\middle|x\right)$$

Proof: $\mathbf{I}_q^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = \mathbf{I}_q^{\alpha}\left\{t^{\gamma-1}\sum_{n=0}^{\infty}\frac{t^n}{\Gamma_q(\alpha n+1)}\right\}(x)$

$$\mathbf{I}_q^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = \sum_{n=0}^{\infty}\frac{1}{\Gamma_q(\alpha n+1)}\{\mathbf{I}_q^{\alpha}t^{n+\gamma-1}\}x$$

Or

$$\mathbf{I}_q^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) =$$

$$\sum_{n=0}^{\infty}\frac{\Gamma_q(n+\gamma)}{\Gamma_q(\alpha+\gamma+1+n)}x^{n+\gamma+\alpha-1} \Rightarrow \mathbf{I}_q^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = x^{\gamma+\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma_q(n+\gamma)}{\Gamma_q(\alpha+\gamma+1+n)}x^n.$$

By Fox–Wright Psi function or just Wright function, we get

$$\mathbf{I}_q^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = x^{\gamma+\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma_q(n+\gamma)\Gamma_q(n+1)}{\Gamma_q(\alpha+\gamma+1+n)}\frac{x^n}{(\mathrm{q;q})_{\mathrm{n}}}$$

Or

$$\mathbf{I}_q^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = x^{\gamma+\alpha-1}\ {}_2\Psi_1\left(\begin{matrix}(\gamma,1)(1,1)\\(\gamma+\alpha+1,\mu)\end{matrix}\middle|x\right)$$

Hence the proof of theorem.

Theorem (2): Let α >0, t $\geq 0, \mu \geq 0, Re(\mu - v) > 0$ and |q|< 1, α, $\mu, v \in C$. Let $\mathbf{D}_{q,x}^{\alpha}$be the Riemann-Liouville fractional derivative operator, then there holds following relation

$$\mathbf{D}_{q,x}^{\alpha}\left\{\{t^{\gamma-1}E_{\alpha}(t:q\}\right\}(x) =$$

Proof:

$$\mathbf{D}_{q,x}^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = \mathbf{D}_{q,x}^{\alpha}\left\{t^{\gamma-1}\sum_{n=0}^{\infty}\frac{t^n}{\Gamma_q(\alpha n+1)}\right\}(x)$$

$$\mathbf{D}_{q,x}^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = \sum_{n=0}^{\infty}\frac{1}{\Gamma_q(\alpha n+1)}\{\mathbf{D}_{q,x}^{\alpha}t^{n+\gamma-1}\}x$$

Or

$$\mathbf{D}_{q,x}^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = \sum_{n=0}^{\infty}\frac{\Gamma_q(n+\gamma)}{\Gamma_q(\gamma-\alpha+1+n)}x^{n+\gamma-\alpha-1}$$

$$\Rightarrow \mathbf{D}_{q,x}^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = x^{\gamma-\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma_q(n+\gamma)}{\Gamma_q(\gamma-\alpha+1+n)}x^n.$$

By Fox–Wright Psi function or just Wright function, we get

$$\mathbf{D}_{q,x}^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = x^{\gamma+\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma_q(n+\gamma)\Gamma_q(n+1)}{\Gamma_q(\gamma-\alpha+1+n)}\frac{x^n}{(\mathrm{q;q})_{\mathrm{n}}}$$

Or

$$\mathbf{D}_{q,x}^{\alpha}\{t^{\gamma-1}E_{\alpha}(t:q\}(x) = x^{\gamma+\alpha-1}\ {}_2\Psi_1\left(\begin{matrix}(\gamma,1)(1,1)\\(\gamma-\alpha+1,\mu)\end{matrix}\middle| x\right)$$

Hence the proof of theorem.

CONCLUSION

The ML- function and its generalization are of fundamental importance in the fractional calculus. It has been shown that the solution of certain fundamental linear differential equations may be expressed in terms of these functions. These functions serve as generalization of the exponential function in the solution of fractional differential equation. Hence these functions play a central role in the fractional calculus. This paper explores various intra relationships of the Ml-function with RL- fractional operator, which will be useful in further analysis. On specializing the parameters we can obtain the corresponding result for exponential function.

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