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Certain expansion formulae involving a basic analogue of I-function

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ABSTRACT

In the present paper, the authors explain certain expansion of the basic analogue of the I-function in relationship with the applications ofq-Leibnitz rule for the Weyl type q-derivatives of a product of two functions. Expansion formulae involving a basic analogue of H-function, Meijer's G-function and MacRobert's E-function have been derived as special cases of the results.

Keywords: Leibnitz rule; Weyl functions; q-operator; Basic analogue of the I-function.

Introduction and Mathematical Preliminaries:

Our translation of real world problems to mathematical expressions relies on calculus, which in turn relies on the differentiation and integration operations of arbitrary order with a sort of misnomer fractional calculus which is also a natural generalization of calculus and its mathematical history is equally long. It plays a significant role in number of fields such as physics, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion, transport theory, probability, elasticity, control theory, engineering mathematics and many others. Fractional calculus like many other mathematical disciplines and ideas has its origin in the quest of researchers for to expand its applications to new fields. This freedom of order opens new dimensions and many problems of applied sciences can be tackled in more efficient way by means of fractional calculus.

The purpose of this paper is to increase the accessibility of different dimensions of q-fractional calculus and generalization of basic hypergeometric functions to the real world problems of engineering, science and economics. Present paper reveals a brief history, definition and applications of basic hypergeometric functions and their generalizations in light of different mathematical disciplines.

The paper is devoted to derive certain expansion formulae for a basic analogue of I-function defined by [1] in terms of Gamma function as follows:

$$I^{m,n}_{q,A_i,B_i;R}\left[z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n}\ (a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m}\ (b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right]=\frac{1}{2\pi w}\int_L\frac{\prod_{j=1}^{m}\Gamma_q(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma_q(1-a_j+\alpha_j s)}{\sum_{i=1}^{R}\left\{\prod_{j=m+1}^{B_i}\Gamma_q(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{A_i}\Gamma_q(a_{ji}-\alpha_{ji}s)\right\}\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}\pi z^s ds \quad (1)$$

Saxena et.al.[2] also defined in terms of Mellin-Barne's integral as follows:

$$I^{m,n}_{A_i,B_i;R}\left[z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n}\ (a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m}\ (b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}G(q^{b_j-\beta_j s})\prod_{j=1}^{n}G(q^{1-a_j+\alpha_j s})\ \pi z^s ds}{\sum_{i=1}^{R}\left\{\prod_{j=m+1}^{B_i}G(q^{1-b_{ji}+\beta_{ji}s})\prod_{j=n+1}^{A_i}G(q^{a_{ji}-\alpha_{ji}s})\right\}G(q^{1-s})\sin\pi s} \quad (2)$$

Where $z \neq 0, 0 < |q| < 1, \omega = \sqrt{-1}$ and
$0 \leq m \leq B_i,\ 0 \leq n \leq A_i,$
$i = 1,2,3, \ldots, R;\ R is finite;$
and $G(q^{\alpha}) = \prod_{m=0}^{\infty}\left\{\left(1-q^{\alpha+n}\right)\right\}^{-1} = \frac{1}{(q^{\alpha};q)_{\infty}}$
Also, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive and a_j, b_j, a_{ji}, b_{ji} are complex numbers.
The contour of integration L runs from $-i\infty$ to $+i\infty$ in such a manner that all the poles of $G(q^{b_j-\beta_j s})$; $1 \leq j \leq m,$ are to the right and those of $G(q^{1-a_j+\alpha_j s})$; $1 \leq j \leq n$ are to its left and are at least some $\varepsilon > 0$ distance away from the contour L. The integral converges if $Re[s\log(z) - \log\sin\pi s] < 0,$ for large values of $|s|$ on the contour that is if $|\arg z| < \pi.$
The q – analogues of H – function is terms of the Mellin – Barne's type basic contour integral is given by *Saxena* [7, 8] as

$$H_{q,A,B}^{m,n}\left[z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,A}\\(b_j,\beta_j)_{1,B}\end{matrix}\right.\right] =$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s})\prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\,\pi z^s ds}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s})\prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}$$

where $G(q^{\alpha}) = \{\prod_{n=0}^{\infty}(1-q^{\alpha+n})\}^{-1} = \frac{1}{(q^{\alpha};q)_{\infty}}$

and $0 \leq m \leq B$, $0 \leq n \leq A$; α_j and β_j are all positive integers, and a_j, b_j are complex numbers, where L is contour of integration running from $-i\infty$ to $i\infty$ in such a manner so that all poles of
$G(q^{b_j-\beta_j s})$ lie to right of the path and $G(q^{1-a_j+\alpha_j s})$ are to the left of the path.
The integral converges if $Re[s\log(z) - \log\sin\pi s\} < 0$, for large values of $|s|$ on the contour L, that is if $|\{\arg(z) - w_2 w_1^{-1}\log|z|\}| < \pi,$ where $|q|<1$, $\log q = -w = -(w_1 + iw_2)$.
The above definition can be used to define the q-analogues of Meijer's G-function as follows:

$$H_{A,B}^{m,n}\left[z;q\left|\begin{matrix}(a,1)_{1,p}\\(b,1)_{1,q}\end{matrix}\right.\right] = G_{A,B}^{m,n}\left[z;q\left|\begin{matrix}a_1,a_2,\ldots,a_p\\b_1,b_2,\ldots,b_q\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_i \frac{\prod_{j=1}^{m} G(q^{b_j-s})\prod_{j=1}^{n} G(q^{1-a_j+s})\,\pi z^s ds}{\prod_{j=m+1}^{q} G(q^{1-b_j+s})\prod_{j=n+1}^{p} G(q^{a_j-s})G(q^{1-s})\sin\pi s}$$

Riemann-Liouville fractional integrals: As defined in [3, 4], the Riemann-Liouville fractional integral is given by:

$$(I_{a+}^{\alpha} f)x = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt,\ (x > a,\ a \epsilon R). \qquad (3)$$

Thus, in general the Riemann-Liouville fractional integrals of arbitrary order for a function f(t), is a natural consequence of the well-knownformula (Cauchy-Dirichlets?) that reduces the calculation of the n-fold primitive of a function f (t) to a single integral of convolution type.
Recently, Yadav et. al. [5] introduces a new q-extension of the Leibnitz rule for the derivatives of a product of two basic functions in terms of a finite q-series involving Weyl type q-derivatives of the functions in the following manner.

$$\underset{z,\ \infty,q}{\overset{\alpha}{D}}\left[U(v)V(z)\right] = \sum_{r=0}^{\infty}\frac{(-1)^r q^{r(r+1)/2}(q^{-\alpha};q)_r}{(q;q)_r}\underset{z\ \ \infty,q}{\overset{\alpha-r}{D}}\left[U(z)\right]\underset{z\ \ \infty,q}{\overset{\alpha}{D}}\left[V(zq^{\alpha-r})\right] \qquad (4)$$

Where U(z) and V(z) are two regular functions and the functional q-differential operator $\underset{z,\ \infty,q}{\overset{\alpha}{D}}(.)$ of Weyl type is given by

$$\underset{z,\ \infty,q}{\overset{\alpha}{D}}[f(z)]=\frac{q^{\frac{-\alpha(\alpha+1)}{2}}}{\Gamma_q(-\alpha)}\int_z^{\infty}(t-z)_{-q-1}f(tq^{1+\alpha})d_q(t) \tag{5}$$

In particular for f(z) = z^{-p} the equation becomes:

$$\underset{z,\ \infty,q}{\overset{\alpha}{D}}\left[z^{-p}\right]=\frac{\Gamma_q(p+\alpha)q^{-\alpha p+\frac{\alpha(1-\alpha)}{2}}z^{-p-\alpha}(1-q)}{\Gamma_q(p)} \tag{6}$$

Main Results:

In this section, we shall establish certain results associated with basic analogue of I-function by assigning suitable values to the functions *z*, and α in the q-Leibnitz rule. The main results to be established are as follows:

$$I^{m+1,n}_{A_i+1,B_i+1;R}\left[P(zq^{\mu})^k;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n}\ (a_{ji},\alpha_{ji})_{n+1,A_i}(\lambda,k)\\(\mu+\lambda,k)(b_j,\beta_j)_{1,m}\ (b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right]=\sum_{r=0}^{\mu}\frac{(-1)^r q^{\frac{r(r+1)}{2}+\lambda r}(q^{-\mu};q)_r(q^{\lambda};q)_{\mu-r}}{(q;q)_r}$$
$$\times I^{m+1,n}_{A_i+1,B_i+1;R}\left[P(zq^{\mu})^k;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}(0,k)\\(r,k)(b_j,\beta_j)_{1,m}\ (b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right] \tag{7}$$

Where

$0\le m\le B,\ 0\le n\le A,\ Re[slog(z)-logsin\pi s]<0, k\ge 0, P\in C.$

Proof. To prove the result,we put

$$U(z)=z^{-\lambda}\ \text{and}\ V(z)=I^{m,n}_{A_i,B_i;R}\left[Pz^k;q\left|\begin{matrix}(a_j,\alpha_j)\ (a_{ji},\alpha_{ji})\\(b_j,\beta_j)\ (b_{ji},\beta_{ji})\end{matrix}\right.\right] \tag{8}$$

in the equation (4) and therefore, we have

$$\underset{z,\ \infty,q}{\overset{\mu}{D}}\left\{z^{-\lambda}I^{m,n}_{A_i,B_i;R}\left[Pz^k;q\left|\begin{matrix}(a_j,\alpha_j)\ (a_{ji},\alpha_{ji})\\(b_j,\beta_j)\ (b_{ji},\beta_{ji})\end{matrix}\right.\right]\right\}=\sum_{r=0}^{\mu}\frac{(-1)^r q^{\frac{r(r+1)}{2}}(q^{-\mu};q)_r}{(q;q)_r}\underset{z,\ \infty,q}{\overset{\mu-r}{D}}\left\{z^{-\lambda}\right\}$$
$$\times\underset{z,\ \infty,q}{\overset{\alpha}{D}}\left\{I^{m,n}_{A_i,B_i;R}\left[P(zq^{\mu-r})^k;q\left|\begin{matrix}(a_j,\alpha_j)\ (a_{ji},\alpha_{ji})\\(b_j,\beta_j)\ (b_{ji},\beta_{ji})\end{matrix}\right.\right]\right\} \tag{9}$$

Now, in view of (1), the L.H.S. of (9) reduces to

$$\underset{z,\ \infty,q}{\overset{\mu}{D}}\left\{z^{-\lambda}I^{m,n}_{A_i,B_i;R}\left[Pz^k;q\left|\begin{matrix}(a_j,\alpha_j)\,(a_{ji},\alpha_{ji})\\(b_j,\beta_j)\,(b_{ji},\beta_{ji})\end{matrix}\right.\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma_q(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma_q(1-a_j+\alpha_j s)\,\pi P^s}{\sum_{i=1}^{R}\left\{\prod_{j=m+1}^{B_i}\Gamma_q(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{A_i}\Gamma_q(a_{ji}-\alpha_{ji}s)\right\}\Gamma_q(s)\Gamma_q(q^{1-s})\sin\pi s}\underset{z,\ \infty,q}{\overset{\mu}{D}}\left\{z^{-(\lambda-ks)}\right\}ds \quad (10)$$

On making use of fractional q-derivative formula in the above equation, we obtain following interesting transformations for the $I_q(.)$ function after certain simplifications.

$$\underset{z,\ \infty,q}{\overset{\mu}{D}}\left\{z^{-\lambda}I^{m,n}_{A_i,B_i;R}\left[Pz^k;q\left|\begin{matrix}(a_j,\alpha_j)\,(a_{ji},\alpha_{ji})\\(b_j,\beta_j)\,(b_{ji},\beta_{ji})\end{matrix}\right.\right]\right\}=\frac{z^{-\lambda-\mu}q^{\mu\lambda+\mu(1-\mu)/2}}{(1-q)^{\mu}}$$

$$\times I^{m+1,n}_{A_i+1,B_i+1;R}\left[P(zq^{\mu})^k;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}(\lambda,k)\\(\mu+\lambda,k)(b_j,\beta_j)_{1,m}\,(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right] \quad (11)$$

Now taking $\lambda = 0$ and replacing μ *by* r and then z *by* $zq^{\mu-r}$ respectively in(11) to get the result

$$\underset{z,\ \infty,q}{\overset{r}{D}}\left\{z^{-\lambda}I^{m,n}_{A_i,B_i;R}\left[P(zq^{\mu-r})^k;q\left|\begin{matrix}(a_j,\alpha_j)\,(a_{ji},\alpha_{ji})\\(b_j,\beta_j)\,(b_{ji},\beta_{ji})\end{matrix}\right.\right]\right\}=\frac{z^{-r}q^{\frac{r(r+1)}{2}-r\mu}}{(1-q)^{r}}$$

$$\times I^{m+1,n}_{A_i+1,B_i+1;R}\left[P(zq^{\mu})^k;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}(0,k)\\(r,k)(b_j,\beta_j)_{1,m}\,(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right] \quad (12)$$

In view of (6), we can easily get the following result

$$\underset{z,\ \infty,q}{\overset{\mu-r}{D}}\left\{z^{-\lambda}\right\}=\frac{\Gamma_q(P+\mu-r)q^{(\mu-r)(1-\mu+r-2P)/2}z^{-P-\mu+r}}{(1-q)^{\mu}} \quad (13)$$

On substituting the values of the expressions involved in (9), from (11), (12) and (13) we arrive at the right hand side of the given theorem (7).

Special cases:

In this section, we shall consider some special cases of the main result and deduce certain expansion formulae for basic analogue of H,G and E- functions as follows:

(i) It is interesting to observe that for taking R = 1, A_i = A, B_i = B, we get well known result of Saxena et.al. [6]

$$H^{m+1,n}_{A+1,B+1}\left[P(zq^{\mu})^k;q\left|\begin{matrix}(a,\alpha)\,(\lambda,k)\\(\mu+\lambda,k)\,(b,\beta)\end{matrix}\right.\right]=\sum_{r=0}^{\mu}\frac{(-1)^r q^{\frac{r(r+1)}{2}+\lambda r}(q^{-\mu};q)_r(q^{\lambda};q)_{\mu-r}}{(q;q)_r}\times H^{m+1,n}_{A+1,B+1}\left[P(zq^{\mu})^k;q\left|\begin{matrix}(a,\alpha)\,(0,k)\\(r,k)(b,\beta)\end{matrix}\right.\right]$$

(14)

(ii) If we set $\alpha = \beta = 1,\ \ and k = 1$, in equation (14), we get the following interesting expansion formula for q-analogue of Meijer's G-function
as,

$$G_{A+1,B+1}^{m+1,n}\left[P\,zq^{\mu};q\left|\begin{matrix}a_1,a_2,...,a_p,\lambda\\ \mu+\lambda,b_1,b_2,...,b_q\end{matrix}\right.\right]=\sum_{r=0}^{\mu}\frac{(-1)^r q^{\frac{r(r+1)}{2}+\lambda r}(q^{-\mu};q)_r(q^{\lambda};q)_{\mu-r}}{(q;q)_r}\times G_{A+1,B+1}^{m+1,n}\left[P\,zq^{\mu};q\left|\begin{matrix}a_1,a_2,...,a_A,0\\ r,b_1,b_2,...,b_B\end{matrix}\right.\right]$$

(15)

(iii) Finally if we set n = 0, and m = B, the result obtained is of q-analogue of MacRobert'sE-function as,

$$E_q\left[B+1;b_j,\mu+\lambda,A+1,A_j,\lambda P\,zq^{\mu}\right]=\sum_{r=0}^{\mu}\frac{(-1)^r q^{\frac{r(r+1)}{2}+\lambda r}(q^{-\mu};q)_r(q^{\lambda};q)_{\mu-r}}{(q;q)_r}\times E_q\left[B+1;b_j,r,A+1,a_j,0,P\,zq^{\mu}\right]$$

(16)

Conclusion

In this paper, we have explored the possibility for derivation of some expansions of basic analogue I- function. The results thus derived are general in character and likely to find certain applications in the theory of hypergeometric functions. Finally we conclude with the remark that the results and the operators proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

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